A Full-order Solution to the Attitude Reset Problem for Kalman Filtering of Attitudes

Rajan Gill*
Swiss Federal Institute of Technology, Zurich, Switzerland

Mark W. Mueller†
University of California, Berkeley, Berkeley, California

Raffaello D'Andrea‡
Swiss Federal Institute of Technology, Zurich, Switzerland

Kalman filtering of states including attitudes poses a challenge due to the constraints of the rotation manifold. One of the standard approaches is to consider a deviation attitude, in the form of a reduced 3-vector parametrization, to some nominal reference attitude, that the Kalman filter tracks. After an update to the statistics of this deviation, a reset step is performed that adjusts the reference attitude. This reset step adjusts the statistics of the deviation, which has commonly been ignored in the literature. This paper presents an algorithm for the case when the deviation is represented as a rotation vector. The adjustment to the mean and covariance after the reset operation is presented, assessed using Monte Carlo sampling, and compared to other approaches used in the literature. The final result may be easily implemented and is computationally inexpensive. Connections and comparisons to the Multiplicative Extended Kalman Filter (MEKF), the Unscented Quaternion Estimator (USQUE), and Lie Group Kalman Filters (LG-KF) are made, with simulations performed on a rigid-body example.

I. Introduction

Determining the attitude of a rigid body in real-time is a problem that was investigated during the space race of the 1960s [1] and is still a topic of research today [2–6]. Because of its computational efficiency and robustness, the Kalman filter is an ideal choice for many applications and has been used in numerous attempts to solve the real-time attitude estimation problem. However, applying the Kalman filter for estimating attitudes is not a trivial problem, as shown in the literature. This is largely attributed to the lack of a vector space structure of the common singularity-free attitude representations such as the Euler symmetric parameters, henceforth referred to as the unit-quaternion, or rotation matrices. As a result, one has to be careful in defining a probabilistic description on such a space, let alone applying Kalman filtering tools. Yet many works [7–14] neglect this fact by directly applying the Kalman filter to e.g. estimating the unit-quaternion parametrization of attitude. Others have indeed pointed out that in doing so, the practical ramifications are an ill-conditioned or even a singular covariance matrix [15]. Other works have circumvented this problem at least in the measurement update using ad-hoc solutions, namely the pre-filtering of measurements [15–17] which are known to produce sub-optimal estimates [17], or purposely adhering to singular and highly nonlinear representations (i.e. Euler angles) to avoid the aforementioned problems [18].

In [19] a norm-constrained Kalman filter was introduced, which has direct applications for estimating the unit-quaternion parametrization for attitudes. This was done for the measurement update by determining the Minimum Mean Squared Error (MMSE) yielding Kalman update gain subject to the norm constraint of the state via Lagrange multipliers. This yields an additive correction to the covariance update, and results in additional theoretical and computational complexity [20].

Another approach is to consider a lower dimensional parametrization of a perturbation attitude, to some deterministic reference attitude, that is to be tracked by the Kalman filter; this approach can be attributed to [21] from 1969. This idea sparked a plethora of research [2, 3, 22–56], some of which have extended this general idea to other Bayesian

*PhD candidate, Institute for Dynamic Systems and Control, ML K35, ETH Zurich, Sonneggstrasse 3, Zurich 8092, Switzerland.
†Assistant Professor, Department of Mechanical Engineering, 5136 Etcheverry Hall, Berkeley, California 94720, USA.
‡Professor, Institute for Dynamic Systems and Control, ML K32.4, ETH Zurich, Sonneggstrasse 3, Zurich 8092, Switzerland.

inferencing schemes such as particle filtering. In the literature, this framework is often referred to as the Multiplicative (Extended) Kalman Filter (MEKF), Indirect Kalman Filter, Sequential Kalman Filter, and Error State Kalman Filter. After every update to this perturbation state (through either prediction via a process model and/or from a measurement update), a so-called reset step is performed where the post-reset reference attitude is adjusted so that the post-reset perturbation state is zero-mean. This reset step ensures that the reference attitude represents the mean attitude of the rigid body (see Section II.B; it is not clear if this interpretation was understood previously in the literature), but more importantly, ensures that the perturbation attitude state stays "small" so as to stay far away from singularities, as no singularity free three-vector parametrization of rotations exists [23].

It has been noted that there is confusion regarding the adjustment to the statistics after the reset step [15, 20]. However, [15] does present the correct adjustment when the deviation attitude is represented by two times the Gibbs-Rodrigues vector*. In [20], the lingering confusion was more formally addressed by formulating the reset problem, presenting a first-order approximation to the exact adjustment for rotation vectors. A similar first-order transformation can be found in [57].

No covariance adjustment (zero-order) and the first-order approximations can still work in practice, as shown in the literature, if the rigid-body of interest does not rotate too quickly with respect to the sampling time, if measurements agree well with the predicted measurements, or through tuning of the problem data. However, there will still be a loss in accuracy. The estimate using the algorithm developed herein using a full-order reset may be expected to be of higher quality in comparison to the previous approaches (examples can be found in Section V).

In Section II, some preliminaries are discussed regarding rotations and probabilities. In particular, a rotation vector parametrization will be used to model the deviation attitude. In Section III, a general estimation problem set-up is discussed, giving context for the attitude reset problem. Specifics regarding rotation kinematics typically encountered in rigid-bodies are also presented, and a corresponding formulation is proposed; this formulation has the nice property of being exact in discrete time. In Section IV, the full-order attitude reset is presented, with Monte Carlo and singular value comparisons to the zero and first-order approaches. If the reader would just like the solution to implement on e.g. the measurement update for the MEKF [23] or the USQUE [26], or to upgrade the attitude reset for the EKF or UKF based on [20], they can jump to this section. In Section V, an example rigid-body is considered, and various EKFs and UKFs are proposed and compared to other approaches in the literature (e.g. MEKF, LG-UKF [4], and USQUE) via simulation. In particular, a computationally-efficient approach is presented for the proposed EKF in Remark 4.

II. Rotations

A. Parameterization and Kinematics

Henceforth rotation matrices will be used to represent attitudes for the purposes of the subsequent derivations, but it is trivial to replace them with the unit-quaternion counterpart for practical implementations. The full-order attitude reset map that will be developed remains the same in either case, as long as the perturbation attitude is represented as a rotation vector.

The set of rotations is denoted by $\mathcal{SO}(3) = \left\{ R \in \mathbb{R}^{3x3} \middle| R^\top R = I_{3\times 3},\ \det(R) = 1 \right\}$. Note that this set is not closed under matrix addition, nor is it commutative under the matrix product. However, it does form a group with compositions of rotations made by their matrix product [58].

Given a three-vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, let its skew-symmetric matrix be

$$[a\times] := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \mathfrak{so}(3) \tag{1}$$

such that the cross product of a and another three-vector b is $a \times b = [a\times]\, b$, and $\mathfrak{so}(3)$ is the set of skew-symmetric matrices contained in $\mathbb{R}^{3x3}$. Similarly, the map that brings a skew-symmetric matrix A to its constituent three-vector is denoted by $[A]^\vee$, that is $[[a\times]]^\vee = a$.

The set of skew-symmetric matrices can parameterize any rotation [59], and by extension so can the three-vectors

*A similar adjustment is presented in [34], but differs by a square-root factor.

(since skew-symmetric matrices and three-vectors are isomorphic). This is realized using the following map

$$\exp : \mathfrak{so}(3) \to \mathcal{SO}(3) \tag{2}$$

$$[a\times] \mapsto \sum_{k\geq 0} \frac{1}{k!}\,[a\times]^k = \begin{cases} I_{3\times 3} + \frac{\sin(\|a\|)}{\|a\|}\,[a\times] + \frac{1-\cos(\|a\|)}{\|a\|^2}\,[a\times]^2 : & \|a\| \neq 0 \\ I_{3\times 3} : & \text{else} \end{cases} \tag{3}$$

where $\|\cdot\|$ denotes the two-norm of its vector argument. The vector a in the above is viewed as a rotation vector. Let $\log(\cdot)$ denote the corresponding inverse mapping given by

$$\log : \mathcal{SO}(3) \to \{\,[a\times] \in \mathfrak{so}(3) \mid \|a\| < \pi\} \tag{4}$$

$$R \mapsto \sum_{k\geq 1} \frac{(-1)^{k+1}}{k}(R - I_{3\times 3})^k = \begin{cases} \frac{\theta(R)}{2\sin(\theta(R))}(R - R^\top) : & \theta(R) \neq 0 \\ 0_{3x3} : & \text{else} \end{cases} \tag{5}$$

where $\theta(R) := \cos^{-1}((\mathrm{Tr}(R) - 1)/2)$. Note that $\log(\exp([a\times])) = a$ only if $\|a\| < \pi$, otherwise wrapping will occur. See e.g. [58–61] for a more thorough treatment.

Consider the following representation for an attitude $R \in \mathcal{SO}(3)$, representing the forward frame rotation from some arbitrary frame to some body-fixed frame of an object

$$R := R^{\mathrm{ref}} \exp([\delta\times]) \tag{6}$$

where δ can be viewed as a perturbation rotation vector to some reference rotation R^{ref}. The dynamics of this rotation vector are given as below, with dependence on time t omitted, [60]:

$$\dot{\delta} = \omega + \frac{1}{2}\,[\delta\times]\,\omega + \frac{2 - \|\delta\|\cot(\|\delta\|/2)}{2\,\|\delta\|^2}\,[\delta\times]^2\,\omega \tag{7}$$

$$=: \Gamma(\delta)^{-1}\omega \tag{8}$$

which hits singularity when $\|\delta\| = 2\pi$, ω is the relative angular velocity of the object expressed in the body frame, and

$$\Gamma : \mathbb{R}^3 \to \mathbb{R}^{3\times 3} \tag{9}$$

$$\delta \mapsto \Gamma(\delta) = \begin{cases} I_{3\times 3} - \frac{1-\cos(\|\delta\|)}{\|\delta\|^2}\,[\delta\times] + \frac{\|\delta\|-\sin(\|\delta\|)}{\|\delta\|^3}\,[\delta\times]^2 : & \|\delta\| \neq 0 \\ I_{3\times 3} : & \text{else} \end{cases} \tag{10}$$

Interestingly, the map $\Gamma(\cdot)$ will also be of use for a Jacobian matrix in an EKF as shown in Section III, and for the attitude reset step in Section IV. Furthermore, $\Gamma(\delta)$ for $\|\delta\| \neq 0$ can be rewritten (see Appendix A[†]) to yield

$$\Gamma(\delta) = \frac{1}{\|\delta\|^2}\left(\delta\delta^\top + [\delta\times]\,(\exp([\delta\times])^\top - I_{3\times 3})\right) \tag{11}$$

which will reduce computation compared to (10) if $\exp([\delta\times])$ is already computed, as will be the case in Section V.B.

It is worth noting that to first order,

$$\Gamma(\delta) \approx I_{3\times 3} - \frac{1}{2}\,[\delta\times] \tag{12}$$

$$\Gamma(\delta)^{-1} \approx I_{3\times 3} + \frac{1}{2}\,[\delta\times] \tag{13}$$

Similarly, the dynamics can be defined directly on the rotation manifold and is given by

$$\dot{R} = R\,[\omega\times] \tag{14}$$

[†]This equivalent map seems to have been discovered recently in [62] using a very different derivation.

B. Mean Attitude

Similar to the discussion in [15], if δ is a random three-vector in (6) with probability distribution function (PDF) $p_\delta(\cdot)$, and thereby R is also a random variable with distribution $p_R(\cdot)$, then naively taking the expectation of (6)

$$\underset{(R)}{\mathrm{E}}[R] = \underset{(\delta)}{\mathrm{E}}\left[R^{\mathrm{ref}}\exp\left([\delta\times]\right)\right] = \int_{\mathbb{R}^3} R^{\mathrm{ref}}\exp\left([x\times]\right)p_\delta(x)\,\mathrm{d}x \tag{15}$$

is to some extent meaningless, as the result will not be a rotation. This comes from the fact that $\mathcal{SO}(3)$ is not a vector space.

Thus, given a probabilistic description of δ, which will be done by the Kalman filter in the subsequent section, it is important to be clear on what the mean attitude is defined to be. First consider the vector space case: the mean $\mu_z \in \mathbb{R}^n$ of a random variable z with PDF $p_z(\cdot)$ is defined to be the one that satisfies [63]

$$0_{n\times1} = \underset{(z)}{\mathrm{E}}[z-\mu_z] = \int_{\mathbb{R}^n}(\bar{z}-\mu_z)p_z(\bar{z})\,\mathrm{d}\bar{z} \tag{16}$$

Similarly, the mean attitude herein is defined as the $\mu_R \in \mathcal{SO}(3)$ that satisfies

$$0_{3\times1} = \underset{(R)}{\mathrm{E}}\left[\left[\log\left(\mu_R^\top R\right)\right]^\vee\right] = \int_{\mathbb{R}^3}\left[\log\left(\mu_R^\top R^{\mathrm{ref}}\exp\left([x\times]\right)\right)\right]^\vee p_\delta(x)\,\mathrm{d}x \tag{17}$$

where the law of the unconscious statistician [64] is used to get the right-hand side. If δ is zero-mean and small in the sense that all its random variates have norm less than π, then

$$\int_{\mathbb{R}^3}\left[\log\left(\left(R^{\mathrm{ref}}\right)^\top R^{\mathrm{ref}}\exp\left([x\times]\right)\right)\right]^\vee p_\delta(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} x p_\delta(x)\,\mathrm{d}x = 0_{3\times1} \tag{18}$$

Thus, in this case, the mean attitude is R^{ref}. This in part motivates the attitude reset step, where the statistics of δ will be transfered in such a way so that after the reset, the mean attitude is represented by R^{ref} and the post-reset perturbation attitude is small in the sense of being zero-mean. When, for example, δ is a Gaussian random three-vector, and thus infinitely many variates have norm equal to or larger than π, then (18) is only an approximation, albeit a good one if the tails are small.

III. Problem Set-up and Solution Approach

Consider the general discrete-time process

$$\xi[k+1],\ R[k+1] = \bar{f}_k(\xi[k], R[k], \eta^{\mathrm{proc}}[k]) \tag{19}$$

where $\eta^{\mathrm{proc}}[k]$ is a zero mean white noise processes, and $\xi[k]$ and $R[k]$ make up the system state (including the attitude) at time step k (both of which are random variables). If the dynamics of the system are inherently continuous (ex. the kinematics of R as given by (14)), then the above can be obtained by some suitable discretization method.

However, it will be natural to work with the local parametrization (6) of the rotation using the rotation vector $\delta[k]$:

$$R[k] := R_k^{\mathrm{ref}}\exp\left([\delta[k]\times]\right) \tag{20}$$

where R_k^{ref} is the deterministic reference attitude at time k (time-indexing of random series will be done with square brackets, and deterministic ones with subscripts). Reasons for doing so are as follows:

- As described in Section II.B, with the rotation vector living in a vector space, it is convenient to describe the statistics of $R[k]$ using the statistics of $\delta[k]$, particularly for computationally efficient Bayesian tracking schemes like the EKF or UKF which presume Euclidean states. This isn't exclusive to rotation vectors though, as e.g. the Gibbs-Rodrigues vector can be used as described in [23]. The Gibbs-Rodrigues vector may also be better at encoding large uncertainties [15]. However, rotation vectors may be more intuitive to work with, having the nice property that $\|\delta\|$ is the angle of rotation.
- The process model (19) will generally involve a numerical integration scheme of the continuous time kinematics (14). Thus to ensure that the numerical integration scheme doesn't violate the constraint that $R[k] \in \mathcal{SO}(3)$, a projection must be performed [65], which adds complexity (although minor in the case of unit-quaternions). This is no

longer an issue when integrating (8), which can be done via the following using the rotation vector composition rule [60] to avoid the singularity in (8):

$$\delta[k+1] = \left[\log\left(\exp\left([\delta[k]\times]\right)\exp\left(\left[\bar{\delta}(\triangle t)\times\right]\right)\right)\right]^{\vee} =: f_{\mathrm{rv}}(\delta[k], \bar{\delta}(\triangle t)) \tag{21}$$

where $\bar{\delta}(\triangle t)$ is the solution to (8), evaluated at the discretization step size $\triangle t$, with initial condition $\bar{\delta}(0) = 0$. If $\omega(t)$ is constant over the time interval $\triangle t$, perhaps due to a measurement from a digital gyroscope sensor, then

$$\delta[k+1] = f_{\mathrm{rv}}(\delta[k], \omega(t_k)\triangle t) \tag{22}$$

i.e. the Euler-integration scheme is exact, and where t_k is the sampled time corresponding to the discrete time step k (i.e. $\omega[k] := \omega(t_k)$).

- For a certain application, it may be more natural to describe the propagation equations using the rotation vector if it is not based on the kinematics (14) (see e.g. [66, 67], which can be extended to 3D).

Thus, by using (20), the model (19) is transformed to the following

$$x[k+1] = f_k(x[k], \eta^{\mathrm{proc}}[k]) \tag{23}$$

with $x[k] := (\xi[k], \delta[k]) \in \mathbb{R}^n$ the new system state, and R_k^{ref} is absorbed into the function $f_k(\cdot,\cdot)$.

Additionally, the following generic measurement model is assumed:

$$y[k] = h_k(x[k], \eta^{\mathrm{meas}}[k]) \tag{24}$$

where again R_k^{ref} is absorbed into the function $h_k(\cdot,\cdot)$.

Now any Bayesian inferencing scheme can be used to track the statistics of the state $x[k]$ using the models (23),(24), such as the Extended Kalman Filter (EKF) or the Unscented Kalman Filter (UKF)[‡].

Remark 1. *If the evolution of the rotation vector is as per the exact kinematics map* (21)*, then the following Jacobians will be needed for the EKF:*

$$\left.\frac{\partial f_{\mathrm{rv}}(\delta,\Delta)}{\partial \delta}\right|_{\delta=0} = \Gamma(\Delta)^{-1}\exp\left([\Delta\times]\right)^{\top} \tag{25}$$

$$\left.\frac{\partial f_{\mathrm{rv}}(\delta,\Delta)}{\partial \Delta}\right|_{\delta=0} = I_{3\times3} \tag{26}$$

See Appendix B for the derivation. For non-zero δ, it would become difficult to construct an analytical expression for the Jacobians.

Next, the notion of an attitude reset is introduced. After the statistics of $\delta[k]$ are updated, via a prior using (23) and/or posterior using (24), then introduce a new deterministic attitude $R_k^{\mathrm{ref,post}}$ and random variable $\delta^{\mathrm{post}}[k]$ such that

$$R[k] = R_k^{\mathrm{ref}}\exp\left([\delta[k]\times]\right) =: R_k^{\mathrm{ref,post}}\exp\left(\left[\delta^{\mathrm{post}}[k]\times\right]\right) \tag{27}$$

with the requirement that $\delta^{\mathrm{post}}[k]$ be zero-mean. Henceforth, the post-reset variables $R_k^{\mathrm{ref,post}}$ and $\delta^{\mathrm{post}}[k]$ replace the pre-reset ones R_k^{ref} and $\delta[k]$, and the Bayesian tracking scheme continues on. The reasons for performing such an attitude reset are as follows:

- If the propagation for $\delta[k]$ is performed using the exact propagation (21), then as mentioned in Remark 1, the Jacobians necessary for the covariance propagation in the EKF are hard to evaluate for non-zero-mean $\delta[k]$.
- Again, if the propagation is according to (21), the dynamics become exceedingly nonlinear for larger $\delta[k]$ (see Appendix B). Thus it is in the best interest for algorithms such as the EKF or UKF to minimize the nonlinearity such that the statistics can be tracked as accurately as possible [68, 73].
- If the propagation is performed according to a numerical integration of the continuous-time kinematics (7), then the reset step must be performed to avoid $\delta[k]$ and its variates becoming large enough to hit singularity.

[‡]A particle filter could also be used. Although, one could elect to work directly on $\mathcal{SO}(3)$ instead of using the local parameterization (6). However, the benefit of working with this local parametrization and employing the attitude reset is having (an approximation to) the mean attitude via R^{ref}, available immediately using (30), and applying the transformation (31) to each particle. Furthermore, the approach herein could be used in a hybrid Kalman-particle filter scheme, see e.g. [68–72].

- If $\delta[k]$ is zero-mean, then as per Section II.B, the mean attitude is available immediately via R_k^{ref}, rather than having to solve for it using the mean attitude definition (17).

In any case, the filter designer may choose to perform the attitude reset whenever they want. However, for the first two points above where the kinematics is used for prediction, the authors recommend performing the attitude reset after every prediction and measurement update step, as will be done in the example of Section V.

As in [20], the problem of how to perform (27) can be summarized succinctly as Problem 1 in the subsequent section. Note that the resulting transformation will also affect the cross-statistics of $\delta[k]$ and $\xi[k]$.

IV. The Attitude Reset Problem and Solution

Problem 1. *Let δ be a random three-vector whose PDF is known and $R^{\text{ref}} \in \mathcal{SO}(3)$ some deterministic attitude. Determine a deterministic $R^{\text{ref,post}} \in \mathcal{SO}(3)$ and random variable $\delta^{\text{post}} \in \mathbb{R}^3$ such that the following are true:*

$$R^{\text{ref}} \exp\left([\delta\times]\right) = R^{\text{ref,post}} \exp\left(\left[\delta^{\text{post}}\times\right]\right) \tag{28}$$

$$\underset{(\delta^{\text{post}})}{\mathrm{E}}\left[\delta^{\text{post}}\right] = 0_{3\times 1} \tag{29}$$

We present a solution to Problem 1 to first-order in δ^{post}, that is, assuming δ^{post} is small, but to full-order in the pre-reset δ. This is a reasonable assumption as part of the problem requirements is for the mean of δ^{post} to be zero, which is some notion of "small". Note that the related work [20] and indirectly [57] present an approximate solution to Problem 1 with the additional assumption that the pre-reset vector δ is small.

Theorem 1. *To first order in δ^{post}, the solution to Problem 1 is*

$$R^{\text{ref,post}} = R^{\text{ref}} \exp\left([\mu\times]\right) \tag{30}$$

$$\delta^{\text{post}} = \Gamma(\mu)\,(\delta - \mu) \tag{31}$$

where $\mu := \underset{(\delta)}{\mathrm{E}}[\delta]$.

Proof. See Appendix C. A similar result, when δ represents two times the vector part of the unit-quaternion, can be found in Appendix D. □

Remark 2. *If the state $x[k]$ as mentioned in Section III is being tracked by an EKF or UKF, then the post-reset covariance is given by*

$$\mathrm{diag}\left(I_{(n-3)\times(n-3)}, \Gamma(\mu)\right) \underset{(x[k])}{\mathrm{Var}}\left[x[k]\right] \mathrm{diag}\left(I_{(n-3)\times(n-3)}, \Gamma(\mu)^\top\right) \tag{32}$$

where $\mathrm{diag}(\cdot)$ *forms a block-diagonal matrix from its arguments.*

It is important to mention that (30) was only known to be true to first-order in the pre-reset δ (and δ^{post}). As it turns out, this is also true to full-order in the pre-reset δ. The previous works that have neglected adjusting the post-reset covariance can be viewed as taking $\Gamma(\mu)$ to zero-order, that is

$$\Gamma_0(\mu) := I_{3\times 3} \approx \Gamma(\mu) \tag{33}$$

Similarly, the works [57], [20] that present the first-order solutions

$$\Gamma_1(\mu) := I_{3\times 3} - [\mu\times]/2 \approx \Gamma(\mu) \tag{34}$$

$$\Gamma_{\exp}(\mu) := \exp\left(-[\mu\times]/2\right) \approx \Gamma(\mu) \tag{35}$$

respectively, are equivalent in the sense that they equal $\Gamma(\mu)$ to at most first-order.

A. Monte Carlo Assessment

The accuracy of the proposed reset (30), (31) can be quantified via Monte Carlo sampling. Henceforth, the prescript ${}_{\text{MC}}(\cdot)$ will be used to denote the Monte Carlo specific variables. The pre-reset ${}_{\text{MC}}\delta$ are sampled from some to-be-defined probability distribution parameterized by some $l, c \in \mathbb{R}^3$. The post-reset ${}_{\mathrm{MC}}\delta^{\mathrm{post}}$ can be computed as suggested by (30), (28):

$$ {}_{\mathrm{MC}}\delta^{\mathrm{post}} := \left[\log\left(\exp\left(\left[-{}_{\mathrm{MC}}\mu(l,c)\times\right]\right)\exp\left(\left[{}_{\mathrm{MC}}\delta\times\right]\right)\right)\right]^{\vee} \tag{36} $$

where

$$ {}_{\mathrm{MC}}\mu(l,c) := \underset{({}_{\mathrm{MC}}\delta|l,c)}{\mathrm{E}}\left[{}_{\mathrm{MC}}\delta\right] \tag{37} $$

Due to the use of the $\log(\cdot)$ operator, it is implied that all variates of ${}_{\mathrm{MC}}\delta^{\mathrm{post}}$ have norm less than π. The post-reset sample mean and variance is computed using (36) to see how well it conforms to (29) and (32) (with $n = 3$). This process is repeated for various probability distributions of ${}_{\mathrm{MC}}\delta$ by randomly generating the parameters l, c, from which we can construct a distribution of errors in the post-reset mean and variance.

1. Error Metrics

Given $l, c \in \mathbb{R}^3$, the PDF for each component $i \in \{0, 1, 2\}$ of ${}_{\mathrm{MC}}\delta$ is given by

$$ p_{{}_{\mathrm{MC}}\delta_i|l_i,c_i}(x) := U(x, c_i - l_i/2, c_i + l_i/2) \tag{38} $$

where

$$ U(x,a,b) := \begin{cases} 1/(b-a): & x \in [a,b] \\ 0: & \text{else} \end{cases} \tag{39} $$

represents a uniform distribution. Using uniform distributions was intentional to avoid the wrapping caused by $\log(\cdot)$, assuming $\|l\|$ is sufficiently small. Note that ${}_{\mathrm{MC}}\mu(l,c) = c$.

Given the parameters l, c, $2^{20} \approx 10^6$ particles are sampled and passed through the map given by (36), from which the post-reset sample mean ${}_{\mathrm{MC}}\delta^{\mathrm{post}}_{\mathrm{s.mean}}(l,c)$ and sample covariance ${}_{\mathrm{MC}}\delta^{\mathrm{post}}_{\mathrm{s,cov}}(l,c)$ are computed to yield the following absolute error metrics

$$ \epsilon_{\mathrm{mean}}(l,c) := \left\|{}_{\mathrm{MC}}\delta^{\mathrm{post}}_{\mathrm{s.mean}}(l,c)\right\| \tag{40} $$

$$ \epsilon_{\mathrm{cov}}(l,c) := \left\|{}_{\mathrm{MC}}\delta^{\mathrm{post}}_{\mathrm{s.cov}}(l,c) - \Gamma\left({}_{\mathrm{MC}}\mu(l,c)\right)\underset{({}_{\mathrm{MC}}\delta|l,c)}{\mathrm{Var}}\left[{}_{\mathrm{MC}}\delta\right]\Gamma\left({}_{\mathrm{MC}}\mu(l,c)\right)^{\top}\right\|_{\mathrm{F}} \tag{41} $$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm of its matrix argument.

2. Ensemble Monte Carlo

We now draw various l and c to create an ensemble sample of mean errors $\epsilon_{\mathrm{mean}}(l,c)$ and covariance errors $\epsilon_{\mathrm{cov}}(l,c)$ as follows:

$$ p_{l_i}(x) := U(x, 0, 1), \quad i = 0, 1, 2 \tag{42} $$

and

$$ c := \begin{bmatrix} r\cos(\theta)\cos(\phi) \\ r\sin(\theta)\cos(\phi) \\ r\sin(\phi) \end{bmatrix} \tag{43} $$

$$ p_{\theta}(x) := U(x, -\pi, \pi) \tag{44} $$

$$ p_{\phi}(x) := \begin{cases} \cos(x)/2: & x \in [-\pi/2, \pi/2] \\ 0: & \text{else} \end{cases} \tag{45} $$

which effectively uniformly samples the sphere with radius r using the parametrization (43).

$2^{20} \approx 10^6$ samples of l, c are drawn for each fixed r and thus 2^{20} values of $\epsilon_{\mathrm{mean}}(l,c)$, $\epsilon_{\mathrm{cov}}(l,c)$ are depicted in Figure 1 for each r. The covariance errors are also compared against the zero-order and first-order versions of the reset map $\Gamma(\cdot)$. The post-reset mean errors are all very small regardless of the size of the pre-reset mean, r, as expected due to

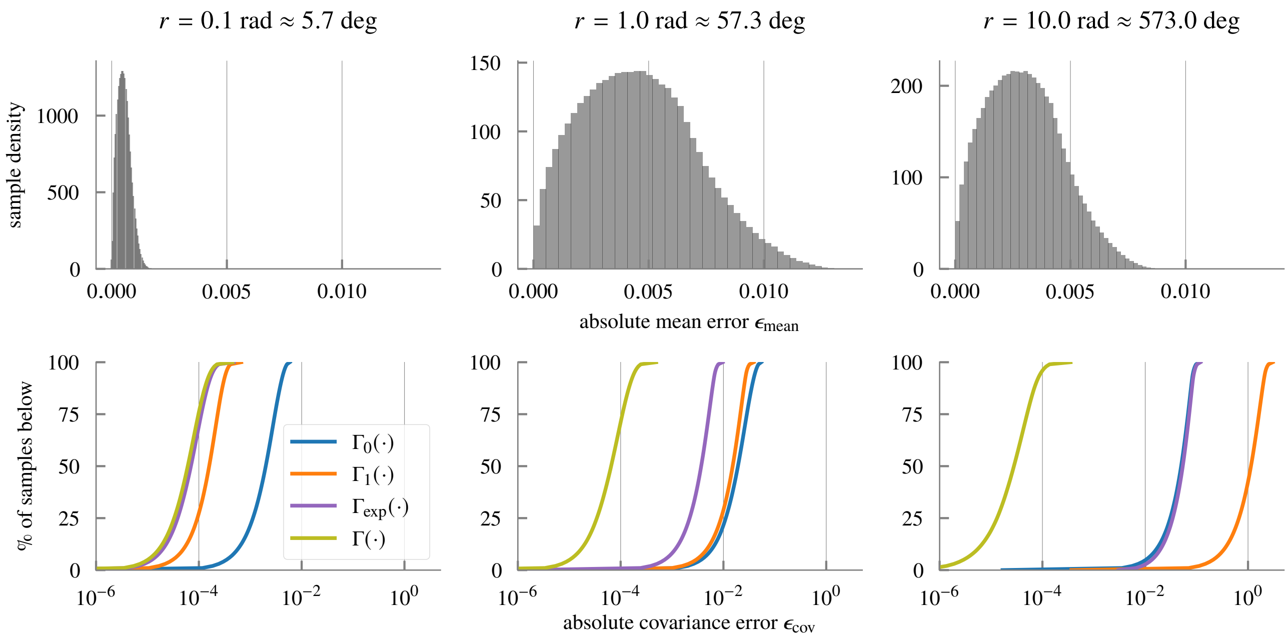

Fig. 1 Monte Carlo assessment: each column represents distinct r values, i.e. the norm of the mean pre-reset rotation vector. The top row shows the absolute mean errors as per (40)**; the bottom row shows the absolute covariance errors as per** (41) **using the approximate and exact $\Gamma(\cdot)$ maps.**

(30) being exact in the pre-reset variable. For example, 95 percent of the samples have mean errors less than 0.0011, 0.0092, and 0.0060, for $r = 0.1$, 1, and 10 rad, respectively.

For rotations, it is only meaningful to report angles modulo π, but, for example, the angle may be larger if integrating the kinematics (8) directly, or if the evolution of the rotation vector is not based on kinematics.

The covariance errors for the full-order map are consistently small and are noticeably better than the zero and first-order versions, especially at the larger pre-reset means. As an example, the 95th percentile errors can be found in Table 1, below.

Table 1 95th percentile of absolute covariance errors

	r [rad]		
	0.1	1.0	10
$\Gamma_0(\cdot)$	0.0044	0.042	0.085
$\Gamma_1(\cdot)$	0.00034	0.028	2.1
$\Gamma_{\exp}(\cdot)$	0.00019	0.0069	0.086
$\Gamma(\cdot)$	0.00017	0.00018	0.000092

For the small r of 0.1, the first-order maps and the full-order map have errors in the same order of magnitude, with the full-order approach outperforming $\Gamma_1(\cdot)$ by a factor of 2. However, this is not the case for $r = 1$ and 10, as the full-order approach outperforms the other approaches by orders of magnitude.

B. Closed Form Comparison

The map $\Gamma(\cdot)$ can be compared to the zero-order and first-order approaches by means of a singular value analysis. Let

$$E_0(\mu) := \Gamma(\mu) - \Gamma_0(\mu) \tag{46}$$

$$= -\frac{1-\cos(\|\mu\|)}{\|\mu\|^2}[\mu\times] + \frac{\|\mu\| - \sin(\|\mu\|)}{\|\mu\|^3}[\mu\times]^2 \tag{47}$$

$$E_1(\mu) := \Gamma(\mu) - \Gamma_1(\mu) \tag{48}$$

$$= \left(-\frac{1-\cos(\|\mu\|)}{\|\mu\|^2} + \frac{1}{2}\right)[\mu\times] + \frac{\|\mu\| - \sin(\|\mu\|)}{\|\mu\|^3}[\mu\times]^2 \tag{49}$$

$$E_{\exp}(\mu) := \Gamma(\mu) - \Gamma_{\exp}(\mu) \tag{50}$$

$$= \left(-\frac{1-\cos(\|\mu\|)}{\|\mu\|^2} + \frac{\sin(\|\mu\|/2)}{\|\mu\|}\right)[\mu\times] + \left(\frac{\|\mu\| - \sin(\|\mu\|)}{\|\mu\|^3} - \frac{1-\cos(\|\mu\|/2)}{\|\mu\|^2}\right)[\mu\times]^2 \tag{51}$$

be the error matrices when $\mu \neq 0$, the difference between the full-order reset matrix $\Gamma(\mu)$ and the respective approximations used in literature. A reasonable scalar metric is the spectral norm (or largest singular value), given by [74]

$$\epsilon_i(\mu) := \|E_i(\mu)\| := \max_{\|x\|=1} \|E_i(\mu)x\| = \sqrt{\lambda_{\max}\left(E_i(\mu)^\top E_i(\mu)\right)}, \quad i \in \{0, 1, \exp\} \tag{52}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of its matrix argument.

Observing that both error matrices have the structure of $c_1\,[\mu\times] + c_2\,[\mu\times]^2$ for some scalars c_1 and c_2, the following fact will prove useful.

Fact 1. *Let $\mu \in \mathbb{R}^3$. The eigenvalue-eigenvector pairs of a matrix of the form*

$$\left(c_1\,[\mu\times] + c_2\,[\mu\times]^2\right)^\top \left(c_1\,[\mu\times] + c_2\,[\mu\times]^2\right) \tag{53}$$

are given by

Eigenvalue	Eigenvector
0	μ
$(c_1\,\|\mu\|)^2 + (c_2\,\|\mu\|^2)^2$	$\mu^\perp$

where $\mu^\perp$ is a vector that satisfies $\mu^T\mu^\perp = 0$.

Proof. Verifiable by direct substitution. □

Thus by Fact 1,

$$\epsilon_0(\mu) = \frac{1}{\|\mu\|}\sqrt{\|\mu\|^2 - 2\|\mu\|\sin(\|\mu\|) - 2\cos(\|\mu\|) + 2} \tag{54}$$

$$\epsilon_1(\mu) = \frac{1}{2\|\mu\|}\sqrt{\|\mu\|^4 + 4\|\mu\|^2\cos(\|\mu\|) - 8\|\mu\|\sin(\|\mu\|) - 8\cos(\|\mu\|) + 8} \tag{55}$$

$$\epsilon_{\exp}(\mu) = 1 - \frac{2\sin(\|\mu\|/2)}{\|\mu\|} \tag{56}$$

Note that these errors do not depend on the direction of the argument μ, rather only its magnitude. Figure 2 shows the plots of the above error terms. Interestingly, there are domains over which the zero-order map is more accurate than the first-order approaches, which is consistent with the results in Figure 1. It is useful to note that for small $\|\mu\|$,

$$\epsilon_0(\mu) \approx \|\mu\|/2 \tag{57}$$

$$\epsilon_1(\mu) \approx \|\mu\|^2/6 \tag{58}$$

$$\epsilon_{\exp}(\mu) \approx \|\mu\|^2/24 \tag{59}$$

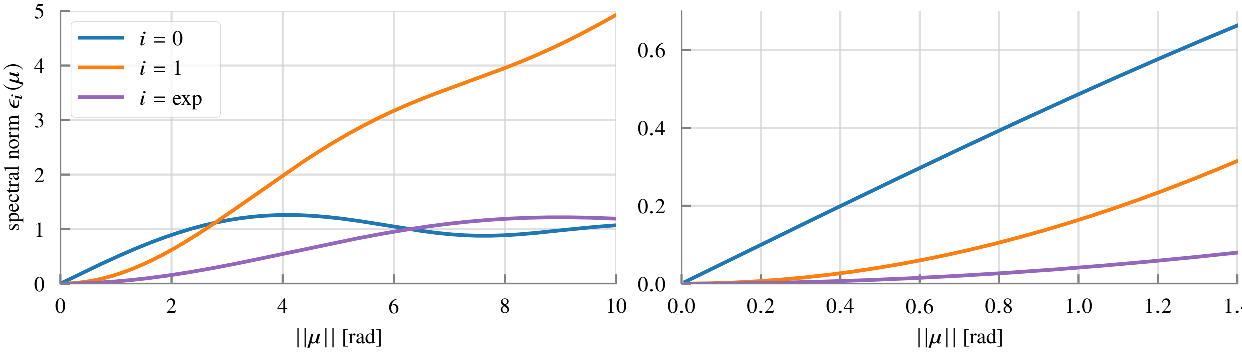

Fig. 2 Spectral norms of the difference between the full-order attitude reset matrix $\Gamma(\mu)$ and its zero/first-order approximations, as defined in (52), over varying magnitudes of the rotation vector argument. A zoomed-in plot is shown on the right.

Loosely speaking, for a Kalman filtering application where the pre-reset mean is not large, that is, in the prediction step either the sampling time is sufficiently small or the motion itself is slow, and corrections due to the measurements are small in the measurement update, the first-order map $\Gamma_1(\cdot)$ will be a very good approximation. Furthermore, additional terms from the series given in (112) can be used for a better approximation. However, when computational expense is of no issue, the full-order map $\Gamma(\cdot)$, which already is of little complexity (and the same complexity as $\Gamma_{\exp}(\cdot)$), should always be used for improved accuracy. For implementations of the EKF/UKF that already use the zero or first-order maps (33)-(35) in the attitude reset, a simple replacement with $\Gamma(\cdot)$ yields an upgrade to a full-order attitude reset.

Remark 3. *Although the EKF approximates the mean and covariance propagations to first and third order, respectively [73], a lower order attitude reset (independently) adds an additional source of error. For example, a measurement update could still lead to a large jump in the estimated state, and thus a full-order covariance reset will be beneficial to help mitigate the overall statistical tracking error; see Section V for an example. This can also be seen in Table 1, where for a pre-reset mean of* 1 rad *the* 95*th percentile covariance errors for* $\Gamma_1(\cdot)$ *are 156 times as large as the* 95*th percentile covariances errors for the full-order* $\Gamma(\cdot)$.

V. Rigid-body Example

A. System

Consider the following rigid body system:

$$\dot{p}(t) = v(t) \tag{60a}$$

$$\dot{v}(t) = R(t)a(t) \tag{60b}$$

$$\dot{R}(t) = R(t)\,[\omega(t)\times] \tag{60c}$$

where $p(t) \in \mathbb{R}^3$ is the position, $v(t) \in \mathbb{R}^3$ is the velocity, $R(t) \in \mathcal{SO}(3)$, $a(t) := (1\,|\cos(t)|\,, 10\,|\sin(t)|\,, 100\,|\cos(t)|)\,\mathrm{m/s^2}$ is the body-frame acceleration vector, and $\omega(t) := (10\,|\sin(t)|\,, 1\,|\cos(t)|\,, 0.1\,|\sin(t)|)\,\mathrm{rad/s}$ is the body-frame angular velocity. The system is simulated numerically using the "embedded Runge-Kutta Prince-Dormand" method [75]. The initial conditions are set to $p(0) := (100, 100, 100)\,\mathrm{m}$, $v(0) := (10, 10, 10)\,\mathrm{m/s}$, and $R(0) := \exp\left([(0, 0, \pi/2)\times]\right)$.

Each estimator will have a noise-free accelerometer sensor that reads $a(t)$, and a gyroscope sensor that reads $\omega(t)$ but corrupted with zero-mean white noise with variance $0.01\,\mathrm{rad^2/s^2}$ per axis. Both sensors are digital and are read at a rate of $1000\,\mathrm{Hz}$. Additionally, a GPS-like sensor provides position readings at $10\,\mathrm{Hz}$ and is also corrupted with zero-mean white noise with variance $100\,\mathrm{m^2}$ per axis. It can be shown that the system is (locally) observable. All noise is generated from Gaussian distributions. All estimators are initialized to believe that the initial position and velocities are zero, and the initial attitude is given by the identity matrix.

The source code for the simulation and all filters tested can be found in Supplementary Material S1.

B. Proposed EKF

The system (60) is discretized using a sample rate of $\triangle t = 0.001$ s. The discretized model uses a combination of a simple Euler integration and the exact rotation propagation (22):

$$\underbrace{\begin{bmatrix} p[k+1] \\ v[k+1] \\ \delta[k+1] \end{bmatrix}}_{=:x[k+1]} = \underbrace{\begin{bmatrix} p[k] + v[k]\triangle t + R_k^{\text{ref}} \exp\left([\delta[k]\times]\right) a_k \triangle t^2/2 \\ v[k] + R_k^{\text{ref}} \exp\left([\delta[k]\times]\right) a_k \triangle t \\ f_{\text{rv}}(\delta[k], (\omega[k] + \eta_\omega[k])\triangle t) \end{bmatrix}}_{=:f_k(x[k], \eta_\omega[k])} \tag{61}$$

with $\underset{(\eta_\omega[k])}{\operatorname{Var}}[\eta_\omega[k]] = 0.01 I_{3\times 3}$, and a_k is the deterministic body-frame acceleration measured at time-step k. Now applying the standard EKF formulas [73], we have for the mean prediction step:

$$\hat{p}_{k+1|k} = \hat{p}_{k|k} + \hat{v}_{k|k}\triangle t + R_k^{\text{ref}} a_k \triangle t^2/2 \tag{62a}$$

$$\hat{v}_{k+1|k} = \hat{v}_{k|k} + R_k^{\text{ref}} a_k \Delta\tau \tag{62b}$$

$$\hat{\delta}_{k+1|k} = f_{\text{rv}}(\hat{\delta}_{k|k}, \overline{\omega_k^{\text{meas}}}\triangle t) = \overline{\omega_k^{\text{meas}}}\triangle t \tag{62c}$$

where $\overline{\omega_k^{\text{meas}}}$ is the realization of the random measurement (note this gives the *a-priori* mean if there is no prior knowledge available on the angular velocity), and in (62c) it is assumed that $\hat{\delta}_{k|k} = 0$, i.e. an attitude reset was just performed before-hand. The covariance propagation is given by

$$\hat{P}_{k+1|k} = A_k \hat{P}_{k|k} A_k^\top + Q \tag{63a}$$

$$A_k = \left[\begin{array}{c|c|c} I_{3\times 3} & \triangle t I_{3\times 3} & -R_k^{\text{ref}}\,[a_k\times]\,\triangle t^2/2 \\ \hline 0_{3\times 3} & I_{3\times 3} & -R_k^{\text{ref}}\,[a_k\times]\,\triangle t \\ \hline 0_{3\times 3} & 0_{3\times 3} & \Gamma(\overline{\omega_k^{\text{meas}}}\triangle t)^{-1} \exp\left(\left[-\overline{\omega_k^{\text{meas}}}\triangle t\times\right]\right) \end{array}\right] \tag{63b}$$

where $\hat{P}_{k+1|k}$ is the estimate of the *a-priori* covariance of the state $x[k+1]$, with $\hat{P}_{0|0} := \operatorname{diag}(I_{6\times 6}, 0.1 I_{3\times 3})$ for this and all following estimators, and $Q = \operatorname{diag}\left(0_{6\times 6}, 0.01\triangle t^2 I_{3\times 3}\right)$. Next the attitude reset is applied as follows:

$$R_{k+1}^{\text{ref}} = R_k^{\text{ref}} \exp\left(\left[\hat{\delta}_{k+1|k}\times\right]\right) \tag{64a}$$

$$\hat{P}_{k+1|k} \leftarrow \operatorname{diag}\left(I_{6\times 6}, \Gamma(\hat{\delta}_{k+1|k})\right) \hat{P}_{k+1|k} \operatorname{diag}\left(I_{6\times 6}, \Gamma(\hat{\delta}_{k+1|k})^\top\right) \tag{64b}$$

$$\hat{\delta}_{k+1|k} \leftarrow 0 \tag{64c}$$

Remark 4. *The propagation equations* (62)-(64) *can be combined, where the* $\Gamma(\cdot)$ *in* (64b) *cancels with the* $\Gamma(\cdot)^{-1}$ *in the Jacobian* A_k*, yielding the following consolidated propagation equations:*

$$\hat{p}_{k+1|k} = \hat{p}_{k|k} + \hat{v}_{k|k}\triangle t + R_k^{\text{ref}} a_k \triangle t^2/2 \tag{65a}$$

$$\hat{v}_{k+1|k} = \hat{v}_{k|k} + R_k^{\text{ref}} a_k \Delta\tau \tag{65b}$$

$$\hat{\delta}_{k+1|k} = 0 \tag{65c}$$

$$R_{k+1}^{\text{ref}} = R_k^{\text{ref}} \exp\left(\left[\overline{\omega_k^{\text{meas}}}\triangle t\times\right]\right) \tag{65d}$$

$$\hat{P}_{k+1|k} = \bar{A}_k \hat{P}_{k|k} \bar{A}_k^\top + \operatorname{diag}\left(I_{6\times 6}, \Gamma(\overline{\omega_k^{\text{meas}}}\triangle t)\right) Q \operatorname{diag}\left(I_{6\times 6}, \Gamma(\overline{\omega_k^{\text{meas}}}\triangle t)^\top\right) \tag{65e}$$

$$\bar{A}_k = \left[\begin{array}{c|c|c} I_{3\times 3} & \triangle t I_{3\times 3} & -R_k^{\text{ref}}\,[a_k\times]\,\triangle t^2/2 \\ \hline 0_{3\times 3} & I_{3\times 3} & -R_k^{\text{ref}}\,[a_k\times]\,\triangle t \\ \hline 0_{3\times 3} & 0_{3\times 3} & \exp\left(\left[\overline{\omega_k^{\text{meas}}}\triangle t\times\right]\right)^\top \end{array}\right] \tag{65f}$$

For computational efficiency, the $\exp\left(\left[\overline{\omega_k^{\text{meas}}}\triangle t\times\right]\right)^\top$ *term in* $\bar{A}_k$ *should be computed first, and then* $\Gamma(\overline{\omega_k^{\text{meas}}}\triangle t)$*, for the transformation on Q, can be computed efficiently using the compact map* (11)*.*

The posterior, or measurement update, is given by

$$\begin{bmatrix}\hat{p}_{k|k}\\ \hat{v}_{k|k}\\ \hat{\delta}_{k|k}\end{bmatrix} = \begin{bmatrix}\hat{p}_{k|k-1}\\ \hat{v}_{k|k-1}\\ 0_{3\times 1}\end{bmatrix} + K_k\left(\overline{y_k^{\text{meas}}} - p_{k|k-1}\right) \tag{66a}$$

$$\hat{P}_{k|k} = (I_{9\times 9} - K_k C)\,\hat{P}_{k|k-1} \tag{66b}$$

where $C = \left[\begin{array}{c|c|c} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3}\end{array}\right]$ and K_k is the Kalman gain using the standard KF formulas [73]. Afterwards, the attitude reset is applied again:

$$R_k^{\text{ref}} \leftarrow R_k^{\text{ref}}\exp\left(\left[\hat{\delta}_{k|k}\times\right]\right) \tag{67a}$$

$$\hat{P}_{k|k} \leftarrow \operatorname{diag}\left(I_{6\times 6}, \Gamma(\hat{\delta}_{k|k})\right)\hat{P}_{k|k}\operatorname{diag}\left(I_{6\times 6}, \Gamma(\hat{\delta}_{k|k})^\top\right) \tag{67b}$$

$$\hat{\delta}_{k|k} \leftarrow 0 \tag{67c}$$

This algorithm is called "ekf-zoh-reset-F" in Figure 3, since it is an EKF based on the exact propagation map (22) for a zero-order-hold assumption on the angular rates, and uses a (F)ull-order attitude reset.

Remark 5. *It can be shown that the proposed EKF herein is equivalent to the LG-EKF of [76] when applied to the proposed model* (61)*. The difference is the derivation, where here it is done directly in the context of attitudes and using the attitude reset perspective. This may make it more accessible to formulate EKFs for other systems, where e.g. some inherent discrete-time process model is used directly for the rotation vector, as mentioned in Section III, allowing for a direct implementation without further derivations.*

C. Other EKFs

One common EKF approach is to model the propagation of δ using an approximation for the rotation vector kinematics (7) [52, 54–56, 77]:

$$\delta[k+1] = \delta[k] + (I_{3\times 3} + [\delta[k]\times]\ /2)\left((\omega[k] + \eta_\omega[k])\triangle t\right) \tag{68}$$

and when performing the attitude reset step in the prediction (64), to use a first-order approximation $\Gamma(\cdot) \approx \Gamma_1(\cdot)$. This results in the mean propagation being the same as (62). The covariance propagation are different in that the $\Gamma(\overline{\omega_k^{\text{meas}}}\triangle t)^{-1}\exp\left(\left[-\overline{\omega_k^{\text{meas}}}\triangle t\times\right]\right)$ term in A_k in (63) is replaced with the approximation $I_{3\times 3} - \left[\overline{\omega_k^{\text{meas}}}\times\right]\triangle t/2$. The measurement updates are the same as (66), and the attitude reset step (67) after the measurement update is applied, again using the first-order approximation $\Gamma(\cdot) \approx \Gamma_1(\cdot)$. This algorithm is called "ekf-euler-reset-1" in Figure 3, as the filter equations are equivalent to the case when an Euler discretization of the continuous-time kinematics of the rotation vector (7) is used, and a first-order approximation for the covariance resets is used (both during the prediction and measurement update).

Another common variant is to take the "ekf-euler-reset-1" filter and omit the post measurement covariance reset step (67b), and is called "ekf-euler-reset-0".

Figure 3 shows the estimator performances for the above EKFs. Both are able to recover the system state, but it is clear that the full-order "ekf-zoh-reset-F" is able to do this more quickly and much more consistently with less spread. What is interesting is that if the initial velocity error were to increase, such that $v(0) = (100, 100, 100)$ m/s, then the "ekf-euler-reset-1" fails completely; see Figure 4. What happens here is that the covariance matrix becomes poorly conditioned, which can happen if δ becomes large in $\Gamma_1(\delta)$ when applying the covariance reset.

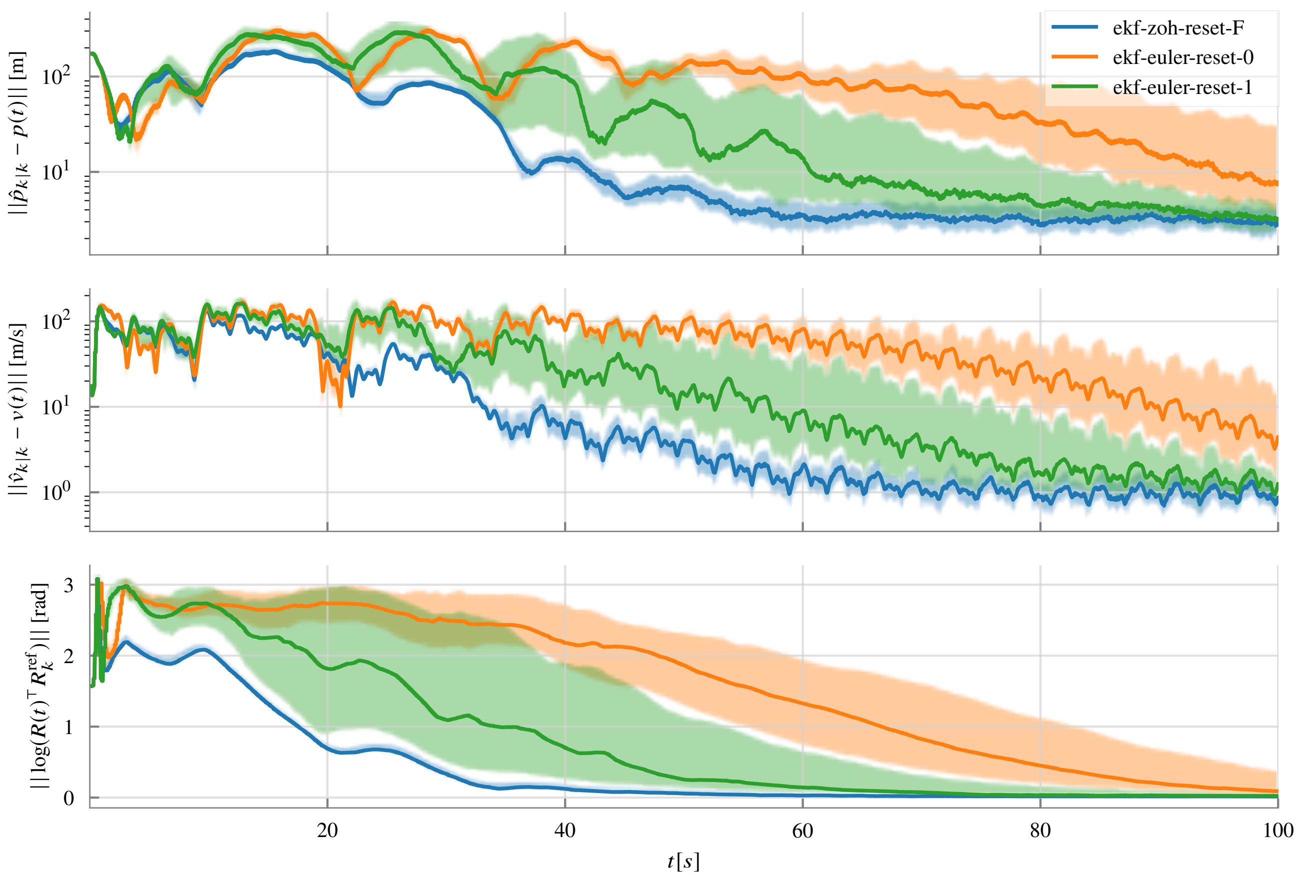

Fig. 3 Position, velocity, and attitude errors of the EKFs described in Sections V.B-V.C. 100 simulations were performed, and the solid lines represent the median values, while the shaded regions represent the 25 to 75 percentile region.

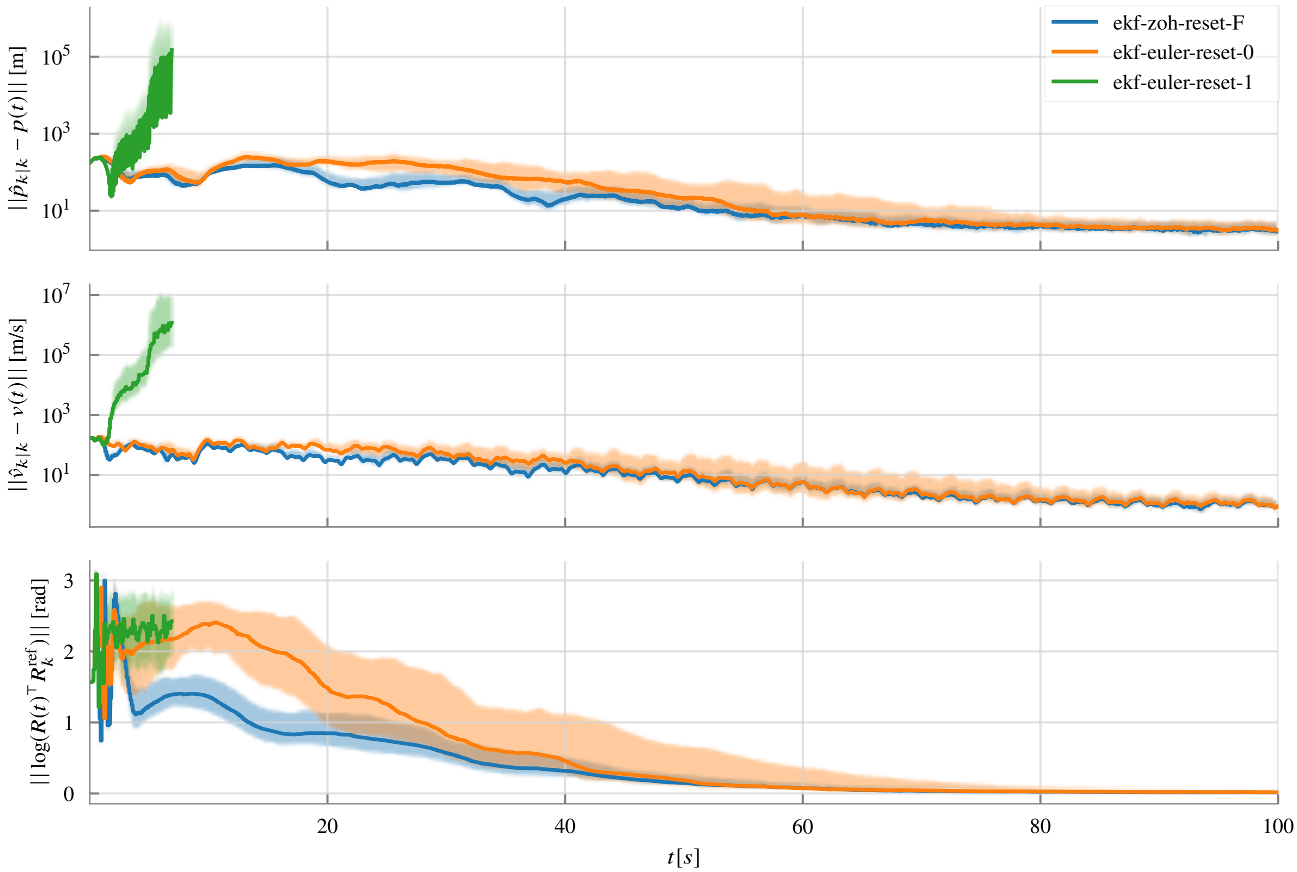

Fig. 4 Position, velocity, and attitude errors of the EKFs described in Sections V.B-V.C, but with $v(0) = (100, 100, 100)$ m/s. 100 simulations were performed, and the solid lines represent the median values, while the shaded regions represent the 25 to 75 percentile region.

D. MEKF [23]

1. Original

The MEKF algorithm of [23], which is formulated in continuous-time, is adapted to fit the example rigid-body system (60). The mean propagation are implemented in discrete-time using (65a)-(65d) for simplicity and consistency. The covariance propagation is adapted to fit the system, yielding

$$\dot{\hat{P}}(t) = F(t)\hat{P}(t) + \hat{P}(t)F(t)^\top + Q_c \tag{69a}$$

$$F(t) = \left[\begin{array}{c|c|c} 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\ \hline 0_{3\times3} & 0_{3\times3} & -R_k^{\text{ref}} \exp\left(\left[\overline{\omega_k^{\text{meas}}}(t-t_k)\times\right]\right)[a_k\times] \\ \hline 0_{3\times3} & 0_{3\times3} & \left[-\overline{\omega_k^{\text{meas}}}\times\right] \end{array}\right] \tag{69b}$$

and $Q_c = \text{diag}\left(0_{6\times6}, 0.01\triangle t I_{3\times3}\right)$ [73]. Since the covariance propagation is formulated in continuous-time, they are implemented using an Euler integration scheme, as done in e.g. [78], with a step-size of $\triangle t_{\text{mekf}}$ (see Algorithm 1), between successive discrete time steps. Computational and implementation aspects favour using such a simple integration scheme, and more accurate integration will yield more accurate results; the parameter $\triangle t_{\text{mekf}}$ will be varied to explore this. Note that no explicit attitude reset step is done in the prediction step, as this is already done implicitly.

Algorithm 1: Euler integration of the MEKF covariance propagation (69)

input: Covariance matrix $\hat{P}_{k|k}$, sampled accelerometer and gyroscope measurements $a_k, \overline{\omega_k^{\text{meas}}}$, the reference attitude R_k^{ref}, integration step size $\triangle t_{\text{mekf}}$, discretization time step $\triangle t$.

output: *a-priori* covariance matrix $\hat{P}_{k+1|k}$.

$\hat{P}_{k+1|k} \leftarrow \hat{P}_{k|k}$;

$N \leftarrow \triangle t / \triangle t_{\text{mekf}}$;

for $j \leftarrow 1$ **to** N **do**

$$F \leftarrow \left[\begin{array}{c|c|c} 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\ \hline 0_{3\times3} & 0_{3\times3} & -R_k^{\text{ref}} \exp\left(\left[\overline{\omega_k^{\text{meas}}}(j-1)\triangle t_{\text{mekf}}\times\right]\right)[a_k\times] \\ \hline 0_{3\times3} & 0_{3\times3} & \left[-\overline{\omega_k^{\text{meas}}}\times\right] \end{array}\right];$$

$\hat{P}_{k+1|k} \leftarrow \hat{P}_{k+1|k} + \left(F\hat{P}_{k+1|k} + \hat{P}_{k+1|k}F^\top + Q_c\right)\triangle t_{\text{mekf}}$;

end

The measurement update is the same as (66). The attitude reset following the update is also the same as (67), except that the covariance reset (67b) does not occur. This filter is called "mekf-0" in Figure 5.

2. Proposed Modifications

In the attitude reset following the measurement update, the step (67b) is performed. This filter is called "mekf-F". Another variant could be to use a first-order approximation $\Gamma(\cdot) \approx \Gamma_1(\cdot)$ in (67b), and is called "mekf-1".

As mentioned in [23], due to the approximations used to derive the filter, the deviation attitude δ in the MEKF can be interpreted to be either a rotation vector as done thus far, two times the Gibbs-Rodrigues vector, or two times the vector part of the unit-quaternion. Thus, following the measurement update (66), two additional versions of the attitude reset (67) are tested: the first is with the $\exp(\cdot)$ map in (67a) replaced with $\exp^{\text{G}}(\cdot)$, mapping two times the Gibbs-Rodrigues vector to the rotation manifold (see e.g. [23, equation (18b)]), and $\Gamma(\cdot)$ in the covariance reset (67b) is replaced with $\Gamma^{\text{G}}(\cdot)$:

$$\Gamma^{\text{G}}(\delta) = \frac{1}{1+\|\delta\|^2/4}\left(I_{3\times3} - [\delta\times]/2\right) \tag{70}$$

the proof of which can be found in the appendix of [15]. This filter variant is called "mekf-G".

Similarly, the other variant is to replace the $\exp(\cdot)$ map in (67a) with $\exp^{\text{qv}}(\cdot)$, mapping two times the vector part of the unit-quaternion to the rotation manifold (see e.g. [23, equation (18d)]), and $\Gamma(\cdot)$ in the covariance reset (67b) is replaced with $\Gamma^{\text{qv}}(\cdot)$:

$$\Gamma^{\text{qv}}(\delta) = \frac{1}{\sqrt{1-\|\delta\|^2/4}}\left(I_{3\times3} + [\delta\times]^2/4\right) - [\delta\times]/2 \tag{71}$$

the proof of which can be found in Appendix D. Note that $\Gamma^{\text{qv}}(\cdot)$ has a singularity and is ill-defined for $\|\delta\| \geq 4$, which can cause problems by creating an ill-conditioned covariance matrix for an application where large resets are to be applied. This filter variant is called "mekf-qv".

The results are shown in Figure 5. The MEKF is sensitive to the integration scheme used to solve (69), as the figure on the left using the larger $\triangle t_{\text{mekf}}$ causes all MEKF variants to perform worse than "ekf-zoh-reset-F". Similar to the previous result in Figure 4, the first-order "mekf-1", and the "mekf-qv", result in poorly conditioned covariance matrices resulting in failure of the estimators, even though the "mekf-0" is still able to recover the state trajectory (on average). The "mekf-G" also performs well, although with more spread than the "ekf-zoh-reset-F".

The covariance propagation for the MEKF was derived by using the continuous-time kinematics and using the quaternion equivalent of (14). This is in contrast to the approach of the "ekf-zoh-reset-F", where the covariance propagation is derived directly in discrete-time using the exact discretization map (22). Thus it is expected that the

MEKF performance (using the correct attitude resets) approaches the performance of the “ekf-zoh-reset-F” in the limit as $\triangle t_{\text{mekf}} \to 0$, although for the MEKF this means an increase in computational complexity [26]. This is exactly what happens for the smaller $\triangle t_{\text{mekf}}$ in the right of Figure 5, where now the full-order variants “mekf-F”, “mekf-G”, and “mekf-qv” perform similar to “ekf-zoh-reset-F”, both in terms of average performance and consistency, while outperforming the “mekf-0” and “mekf-1” variants.

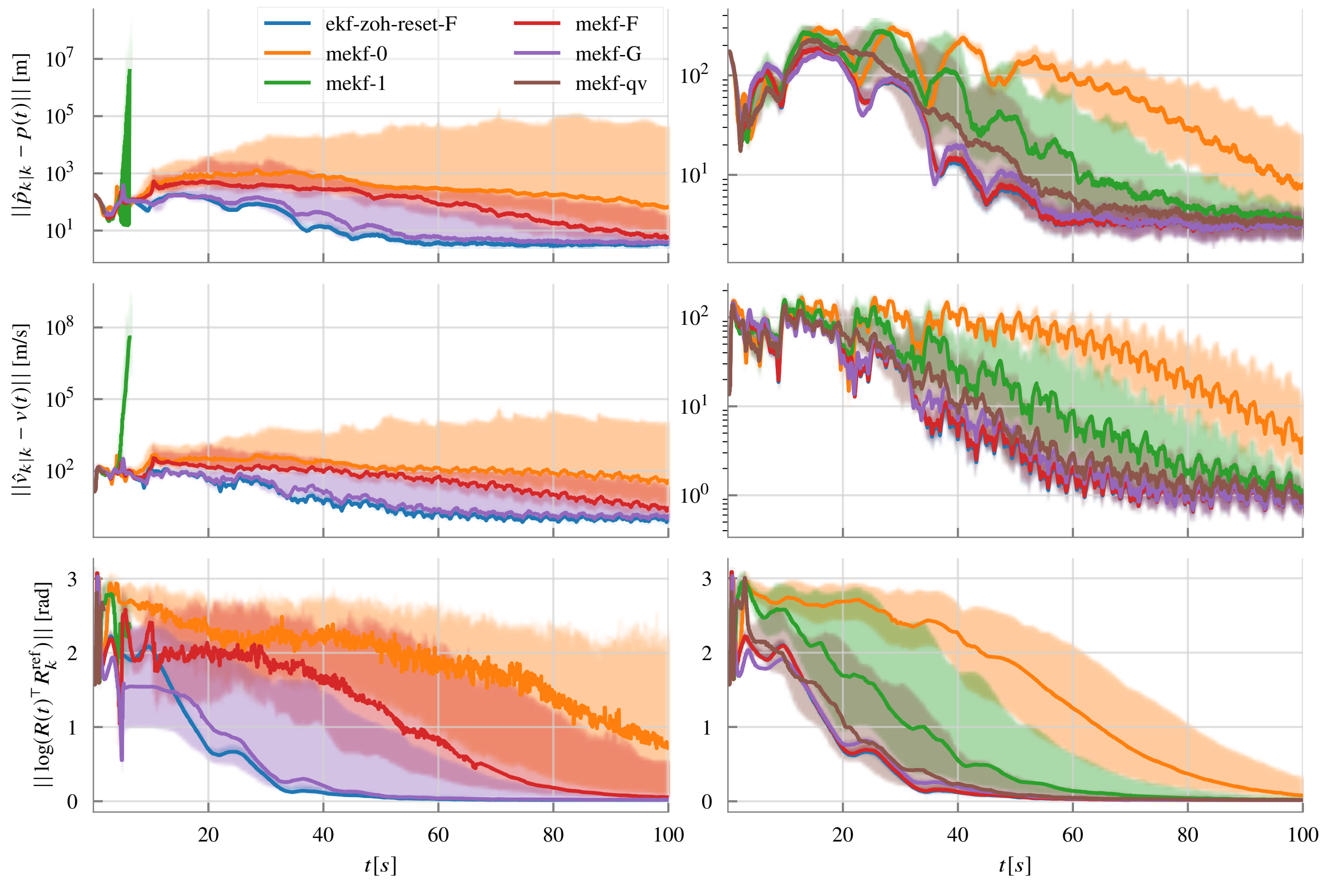

Fig. 5 Position, velocity, and attitude errors of the MEKF using various attitude resets as described in Section V.D. 100 simulations were performed, and the solid lines represent the median values, while the shaded regions represent the 25 to 75 percentile region. The left figure is for $\triangle t_{\text{mekf}} = \triangle t = 0.001$ s; the right figure is for $\triangle t_{\text{mekf}} = \triangle t/100 = 0.00001$ s, taking roughly 100x the computation time. The “ekf-zoh-reset-F” from Section V.B is shown for reference.

E. Proposed UKF

Begin by generating the sigma-points $x^{(i)}_{k|k} = \left(p^{(i)}_{k|k}, v^{(i)}_{k|k}, \delta^{(i)}_{k|k}\right)$, $i = 0, \ldots, 2n$ (with $n = 9$ in this case), for the system (61), using the state covariance $\hat{P}_{k|k}$ and any sigma-point generation method [73, 79]. Since the process noise is non-additive in the dynamics map (61), the system state can be augmented to include the process noise, yielding 6 additional sigma points in this case. However for simplicity and consistency with the other UKFs discussed later, the process noise will be incorporated using the standard EKF approach. The sigma-points are propagated forward:

$$x^{(i)}_{k+1|k} := f_k(x^{(i)}_{k|k}, 0), \quad i = 0, \ldots, 2n \tag{72}$$

The mean *a-priori* state is then taken to be the weighted sum

$$\hat{x}_{k+1|k} := \sum_{i=0}^{2n} w_m^{(i)} x_{k+1|k}^{(i)} \tag{73}$$

where $w_m^{(i)}$ are weighting coefficients [73, 79]. Similarly, the *a-priori* covariance is estimated

$$\hat{P}_{k+1|k} := \sum_{i=0}^{2n} w_c^{(i)} \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right) \left(x_{k+1|k}^{(i)} - \hat{x}_{k+1|k} \right)^\top + Q \tag{74}$$

where $Q = \operatorname{diag}\left(0_{6\times6}, 0.01\triangle t^2 I_{3\times3}\right)$ is the same as before, and $w_c^{(i)}$ are weighting coefficients [73, 79]. Now proceed with the post-prediction attitude reset (64)[§].

Since the measurement model is linear in the system state, the measurement update is the same as (66). The post-measurement attitude reset (67) is applied. This filter is called "ukf-zoh-reset-F" in Figure 6.

F. LG-UKF [4]

The LG-UKF algorithm of [4] is adapted to fit the example rigid-body system. The propagation equations are similar to the previous subsection, except that the sigma-points for the rotation are computed directly on $\mathcal{SO}(3)$ as follows:

$$R_{k+1|k}^{(i)} = R_k^{\text{ref}} \exp\left(\left[\delta_{k|k}^{(i)} \times \right] \right) \exp\left(\left[\overline{\omega_k^{\text{meas}}} \triangle t \times \right] \right), \quad i = 0, \ldots, 2n \tag{75}$$

The mean rotation R_{k+1}^{ref} is computed numerically via [4, Algorithm 1], written in Appendix E using the convention of this paper. The covariance propagation is computed as follows:

$$\hat{P}_{k+1|k} := \sum_{i=0}^{2n} w_c^{(i)} d^{(i)} d^{(i)\top} + Q \tag{76}$$

$$d^{(i)} := \left(p_{k+1|k}^{(i)} - \hat{p}_{k+1|k},\ v_{k+1|k}^{(i)} - \hat{v}_{k+1|k},\ \left[\log\left(\left(R_{k+1}^{\text{ref}} \right)^{-1} R_{k+1|k}^{(i)} \right) \right]^\vee \right) \tag{77}$$

No attitude reset is performed, as this is done implicitly. Furthermore, note that the process noise covariance matrix Q is not transformed, where the approximation $\Gamma(\overline{\omega_k^{\text{meas}}}\triangle t) \approx \Gamma_{\exp}(\overline{\omega_k^{\text{meas}}}\triangle t)$ is used, which as shown in Section IV.B, is a very good approximation for small $\overline{\omega_k^{\text{meas}}}\triangle t$. This combined with the fact that Q is diagonal (as in [4] and in this case), results in $\operatorname{diag}\left(I_{6\times6}, \Gamma_{\exp}(\overline{\omega_k^{\text{meas}}}\triangle t)\right) Q \operatorname{diag}\left(I_{6\times6}, \Gamma_{\exp}(\overline{\omega_k^{\text{meas}}}\triangle t)\right)^\top = Q$. However, this will not be true for non-isotropic process noise.

The measurement update is the same as before in (66) - (67). This filter is called "lg-ukf" in Figure 6.

G. USQUE [26]

1. Original

The USQUE, presented in [26], is adapted to fit the example rigid-body system. As originally presented using the generalized Rodrigues parameters, two times the Gibbs-Rodrigues vector will be used herein. Thus the $\exp(\cdot)$ map in the system model (61), used to propagate the position and velocity sigma-points, is replaced with the $\exp^{\text{G}}(\cdot)$ map. The rotation sigma-points are propagated as follows:

$$R_{k+1|k}^{(i)} := R_k^{\text{ref}} \exp^{\text{G}}\left(\left[\delta_{k|k}^{(i)} \times \right] \right) \exp\left(\left[\overline{\omega_k^{\text{meas}}} \triangle t \times \right] \right), \quad i = 0, \ldots, 2n \tag{78}$$

However, based on equation (36) and (37a) of [26], it is presumed that $R_{k+1|k}^{(0)}$ represents the mean *a-priori* attitude, and thus $R_{k+1}^{\text{ref}} := R_{k+1|k}^{(0)}$. Since $\hat{\delta}_{k|k}^{(0)} = 0$, this essentially means that the USQUE does an EKF approximation for the mean

[§]The attitude reset presented herein is true to first-order in the post-reset deviation attitude. Thus, for a situation where large uncertainty in the attitude is expected, the LG-UKF may perform better. Nevertheless, the proposed UKF can be viewed as an upgrade to the approach in [20, 53].

attitude propagation (see e.g. (65d)). The covariance propagation is computed as follows:

$$\hat{P}_{k+1|k} := \sum_{i=0}^{2n} w_c^{(i)} d^{(i)} d^{(i)\top} + Q \tag{79}$$

$$d^{(i)} := \left(p_{k+1|k}^{(i)} - \hat{p}_{k+1|k},\ v_{k+1|k}^{(i)} - \hat{v}_{k+1|k},\ \left[\log^{\mathrm{G}} \left(\left(R_{k+1}^{\mathrm{ref}} \right)^{-1} R_{k+1|k}^{(i)} \right) \right]^{\vee} \right) \tag{80}$$

where $\log^{\mathrm{G}}(\cdot)$ maps the rotation element on $\mathcal{SO}(3)$ back to two-times the Gibbs-Rodridgues vector (see e.g. [26, equation (20)]). Thus the USQUE is very similar to the LG-UKF, except that the mean attitude propagation is done using an EKF approximation, and the deviation attitude δ is viewed as two times the Gibbs-Rodrigues vector (or generalized Rodrigues parameters). Again, there is no transformation applied to the process noise covariance matrix Q (note that [26] does present a method for integrating Q_c using a trapezoidal integration scheme and under the assumption that $\omega \triangle t$ is small, which would result in adding a factor $Q/2$ to $\hat{P}_{k|k}$ when generating the sigma-points in the propagation step, and replacing Q in (79) with the same factor, in this case). The measurement update is the same as before in (66), and the attitude reset (67) with $\exp(\cdot)$ replaced with $\exp^{\mathrm{G}}(\cdot)$, except the covariance reset (67b) does not occur. This filter is called "usque-0" in Figure 6.

2. Proposed Modification

One obvious modification is to perform the covariance adjustment after the measurement update (67b), with $\Gamma(\cdot)$ replaced with the Gibbs version $\Gamma^{\mathrm{G}}(\cdot)$ given in (70), and will be called "usque-G". Another approach is to additionally perform a UKF mean propagation, which can be done by solving for the propagated mean either numerically using an approach like the LG-UKF, or using an attitude reset approach like the proposed UKF. This, however, will not be explored in this paper.

The results for the above UKFs are shown in Figure 6. The sigma points are generated using the scheme outlined in [73, Section 14.2], and applying the scaling from [79] with $\alpha = 1$ and $\beta = 2$. The "ekf-zoh-reset-F", "ukf-zoh-reset-F", "lg-ukf", and the "usque-G" all behave similarly, and perform better than "usque-0" both in terms of average and consistency of errors.

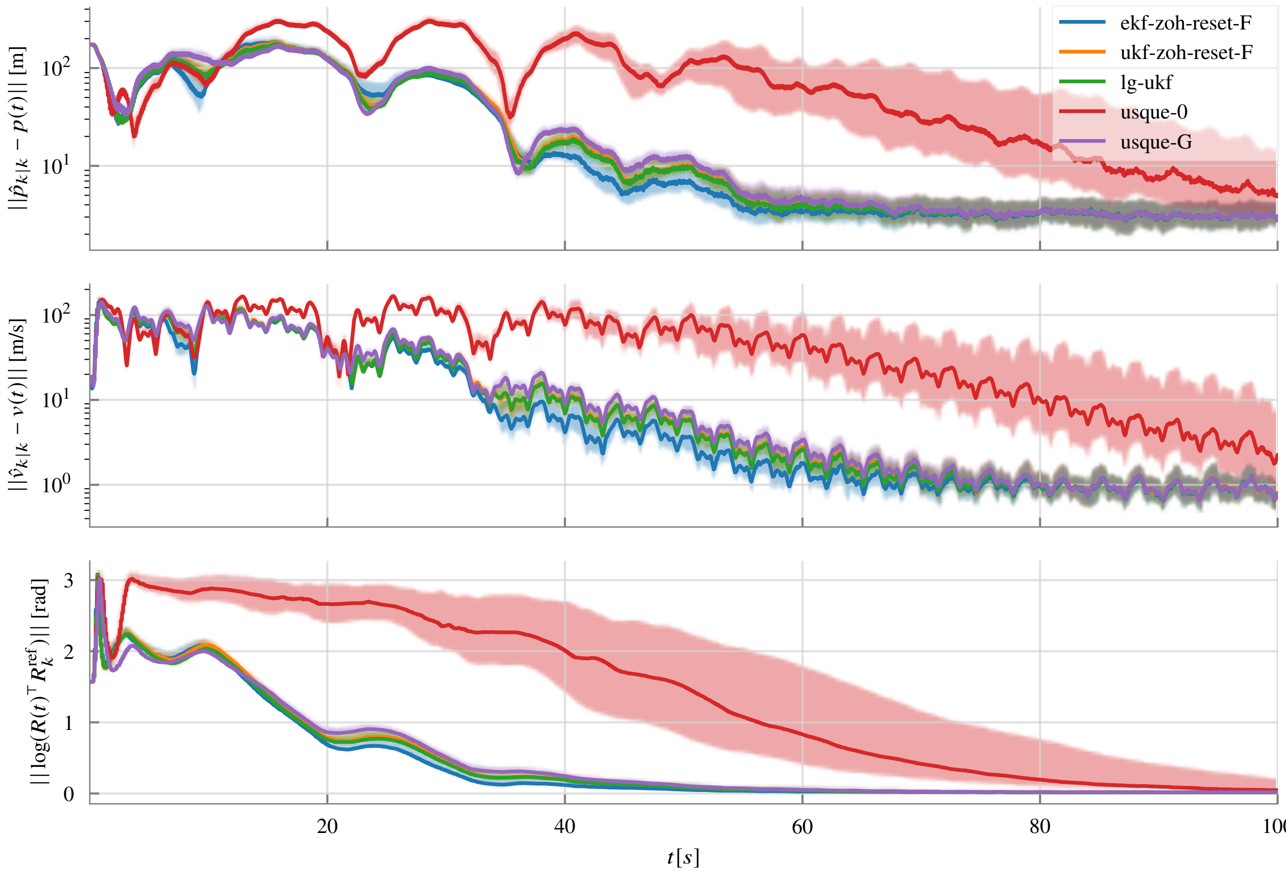

Fig. 6 Position, velocity, and attitude errors of the UKFs described in Sections V.E-V.G. 100 simulations were performed, and the solid lines represent the median values, while the shaded regions represent the 25 to 75 percentile region. The "ekf-zoh-reset-F" from Section V.B is shown for reference.

VI. Conclusions

Theorem 1 establishes the relationship between the pre-reset and post-reset attitude error states to full-order in the former and first-order in the latter. This can be used in an EKF and UKF for a general problem set-up involving attitudes. The proposed full-order reset step and the proposed EKF and UKF algorithms are computationally efficient and easy to implement. The full-order reset also offers a free upgrade to any of the zero or first-order variants used in practice. Comparisons are also made with the MEKF and the USQUE using various attitude reset schemes, showing that a full-order reset can be beneficial.

Appendix

A. Equivalent $\Gamma(\cdot)$ map

By [74, Fact 3.5.25], $[a\times]^2 = aa^\top - \|a\|^2 I_{3\times3}$, thus (10) can be rewritten to

$$\Gamma(a) = \frac{\sin(\|a\|)}{\|a\|} I_{3\times3} - \frac{1-\cos(\|a\|)}{\|a\|^2}[a\times] + \frac{\|a\| - \sin(\|a\|)}{\|a\|^3} aa^\top \tag{81}$$

Rearranging,

$$\Gamma(a) = \frac{1}{\|a\|^2} aa^\top + \frac{\sin(\|a\|)}{\|a\|^3}\left(-aa^\top + \|a\|^2 I_{3\times3}\right) - \frac{1-\cos(\|a\|)}{\|a\|^2}[a\times] \tag{82}$$

Again by [74, Fact 3.5.25], $[a\times]^3 = -\|a\|^2\,[a\times]$, thus

$$\Gamma(a) = \frac{1}{\|a\|^2}aa^\top + \frac{\sin(\|a\|)}{\|a\|^3}\left(-\,[a\times]^2\right) + \frac{1-\cos(\|a\|)}{\|a\|^4}\,[a\times]^3 \tag{83}$$

$$= \frac{1}{\|a\|^2}aa^\top + \frac{1}{\|a\|^2}\,[a\times]\left(-\frac{\sin(\|a\|)}{\|a\|}\,[a\times] + \frac{1-\cos(\|a\|)}{\|a\|^2}\,[a\times]^2\right) \tag{84}$$

$$= \frac{1}{\|a\|^2}aa^\top + \frac{1}{\|a\|^2}\,[a\times]\,(\exp([-a\times]) - I_{3\times3}) \tag{85}$$

$$= \frac{1}{\|a\|^2}aa^\top + \frac{1}{\|a\|^2}\,[a\times]\,\left(\exp([a\times])^\top - I_{3\times3}\right) \tag{86}$$

B. Derivation of (25)**-**(26)

Lemma 1. *If* $X,\ Y,\ Z \in \mathfrak{so}(3)$ *with*

$$\exp(Z) = \exp(X)\exp(Y) \tag{87}$$

then,

$$Z = X + \frac{-\mathrm{ad}_X}{\exp(-\mathrm{ad}_X) - 1}(Y) + O\left(Y^2\right) \tag{88}$$

$$= Y + \frac{\mathrm{ad}_Y}{\exp(\mathrm{ad}_Y) - 1}(X) + O\left(X^2\right) \tag{89}$$

where ad *denotes the matrix Lie-group adjoint mapping* $\mathrm{ad}_X(Y) := XY - YX$*, the* k*-th power of* ad_X *denotes its* k*-th iterate with* $\mathrm{ad}_X^0(Y) = Y$*, and* $\exp(A) = \sum_{k\geq 0}\frac{1}{k!}A^k$*. See [61] for a more thorough treatment.*

Proof. Equation (88) can be obtained by performing straightforward manipulations to the integral form of the Baker-Campbell-Hausdorff (BCH) formula (5.8) of [61].

For (89), start with:

$$\exp(Z(t)) = \exp(tX)\exp(Y) \tag{90}$$

The goal is to determine $Z(1)$, with $Z(0) = Y$, minding the slight abuse of notation with Z versus $Z(t)$. Taking the derivative, from [61, equation (5.11)] we have

$$\frac{\mathrm{d}\exp(Z(t))}{\mathrm{d}t} = \exp(Z(t))\frac{1-\exp(-\mathrm{ad}_{Z(t)})}{\mathrm{ad}_{Z(t)}}\left(\dot{Z}(t)\right) = X\exp(tX)\exp(Y) \tag{91}$$

$$= X\exp(Z(t)) \tag{92}$$

Thus,

$$\frac{1-\exp(-\mathrm{ad}_{Z(t)})}{\mathrm{ad}_{Z(t)}}\left(\dot{Z}(t)\right) = \exp(-Z(t))X\exp(Z(t)) \tag{93}$$

$$= \mathrm{Ad}_{\exp(-Z(t))}(X) \tag{94}$$

$$= \exp(-\mathrm{ad}_{Z(t)})(X) \tag{95}$$

where [61, Proposition 3.35] was used. Thus,

$$\dot{Z}(t) = \frac{\mathrm{ad}_{Z(t)}}{1-\exp(-\mathrm{ad}_{Z(t)})}\exp(-\mathrm{ad}_{Z(t)})(X) \tag{96}$$

$$= \frac{\mathrm{ad}_{Z(t)}}{\exp(\mathrm{ad}_{Z(t)}) - 1}(X) \tag{97}$$

$$= \frac{\log(\exp(\mathrm{ad}_{tX})\exp(\mathrm{ad}_Y))}{\exp(\mathrm{ad}_{tX})\exp(\mathrm{ad}_Y) - 1}(X) \tag{98}$$

$$= \bar{g}(\exp(\mathrm{ad}_{tX})\exp(\mathrm{ad}_Y))(X) \tag{99}$$

where [61, Theorem 3.28] was used, $\log(A) := \sum_{k\geq 1} \frac{(-1)^{k+1}}{k}(A-I)^k$, and $\bar{g}(A) := \sum_{k\geq 0} \frac{(-1)^k}{k+1}(A-I)^k$. Thus, an alternate integral form of the BCH is

$$Z = Y + \int_0^1 \bar{g}(\exp(\mathrm{ad}_{tX})\exp(\mathrm{ad}_Y))(X)\mathrm{d}t \tag{100}$$

and thus by performing straightforward manipulations yields (89). $\square$

Using (89), (21) becomes

$$f_{\mathrm{rv}}(\delta, \Delta) = \Delta + \left[\underbrace{\frac{\mathrm{ad}_{[\Delta\times]}}{\exp(\mathrm{ad}_{[\Delta\times]}) - 1}([\delta\times])}_{=:[\xi\times]} + O\left([\delta\times]^2\right) \right]^{\vee} \tag{101}$$

where $\Delta := \bar{\delta}(\triangle t)$ for brevity. Resolving the middle term,

$$\left(\frac{\exp(\mathrm{ad}_{[\Delta\times]}) - 1)}{\mathrm{ad}_{[\Delta\times]}}\right)[\xi\times] = [\delta\times] \tag{102}$$

$$\left(\exp(\mathrm{ad}_{[\Delta\times]})\frac{1-\exp(-\mathrm{ad}_{[\Delta\times]})}{\mathrm{ad}_{[\Delta\times]}}\right)[\xi\times] = \tag{103}$$

Note that

$$\left[\mathrm{ad}_{[x\times]}([y\times])\right]^{\vee} = [([x\times]\ [y\times] - [y\times]\ [x\times])]^{\vee} \tag{104}$$

$$= [x\times]\ y \tag{105}$$

$$\Leftrightarrow \left[\mathrm{ad}^k_{[x\times]}([y\times])\right]^{\vee} = [x\times]^k\ y \tag{106}$$

where the second equality comes from [74, Fact 3.5.25]. Thus,

$$\left[\left(\exp(\mathrm{ad}_{[\Delta\times]})\frac{1-\exp(-\mathrm{ad}_{[\Delta\times]})}{\mathrm{ad}_{[\Delta\times]}}\right)[\xi\times]\right]^{\vee} = \delta \tag{107}$$

$$\left(\exp([\Delta\times])\frac{1-\exp(-[\Delta\times])}{[\Delta\times]}\right)\xi = \tag{108}$$

$$\exp([\Delta\times])\Gamma(\Delta)\xi = \tag{109}$$

$$\Leftrightarrow \xi = \Gamma(\Delta)^{-1}\exp(-[\Delta\times])\delta \tag{110}$$

where for the step from (108) to (109) the following is used[¶]:

[¶]A similar relationship can also be found in [80, 81] without derivation.

$$\frac{1-\exp(-[\Delta\times])}{[\Delta\times]} = \frac{I-\sum_{k\geq 0}(-[\Delta\times])^k/k!}{[\Delta\times]} \tag{111}$$

$$= \sum_{k\geq 0} \frac{(-1)^k}{(k+1)!}[\Delta\times]^k \tag{112}$$

$$= \sum_{k\geq 0} \left(\frac{(-1)^{2k}}{(2k+1)!}[\Delta\times]^{2k} + \frac{(-1)^{2k+1}}{(2k+2)!}[\Delta\times]^{2k+1} \right) \tag{113}$$

$$= \left(I_{3\times 3} + \sum_{k\geq 1} \frac{(-1)^{3k-1}\|\Delta\|^{2k-2}}{(2k+1)!}[\Delta\times]^2 + \sum_{k\geq 0} \frac{(-1)^{3k+1}\|\Delta\|^{2k}}{(2k+2)!}[\Delta\times] \right) \tag{114}$$

$$= \left(I_{3\times 3} + \sum_{k\geq 0} \frac{(-1)^{k}\|\Delta\|^{2k}}{(2k+3)!}[\Delta\times]^2 + \sum_{k\geq 1} \frac{(-1)^{k}\|\Delta\|^{2k-2}}{(2k)!}[\Delta\times] \right) \tag{115}$$

$$= \left(I_{3\times 3} + \frac{1}{\|\Delta\|^2}\left(1 - \frac{1}{\|\Delta\|}\sum_{k\geq 0} \frac{(-1)^{k}\|\Delta\|^{2k+1}}{(2k+1)!}\right)[\Delta\times]^2 + \frac{1}{\|\Delta\|^2}\left(\sum_{k\geq 0} \frac{(-1)^{k}\|\Delta\|^{2k}}{(2k)!} - 1\right)[\Delta\times] \right) \tag{116}$$

$$= \left(I_{3\times 3} + \frac{1}{\|\Delta\|^2}\left(1 - \frac{1}{\|\Delta\|}\sin(\|\Delta\|)\right)[\Delta\times]^2 + \frac{1}{\|\Delta\|^2}(\cos(\|\Delta\|)-1)[\Delta\times] \right) \tag{117}$$

$$= \Gamma(\Delta) \tag{118}$$

where the following is used to go from steps (113) to (114):

$$[x\times]^{2k+1} = (-1)^k \|x\|^{2k}[x\times], \quad k = 0,1,2,\dots \tag{119}$$

$$[x\times]^{2k} = (-1)^{k-1} \|x\|^{2k-2}[x\times]^2, \quad k = 1,2,\dots \tag{120}$$

Thus,

$$f_{\mathrm{rv}}(\delta,\Delta) = \Delta + \Gamma(\Delta)^{-1}\exp([\Delta\times])^\top \delta + \left[O\left([\delta\times]^2\right)\right]^\vee \tag{121}$$

from which (25)-(26) is obtained. Note that from (121), the non-linearity in the map increases as δ increases.

C. Solution to Problem 1

Since $R^{\mathrm{ref}}, R^{\mathrm{ref,post}} \in \mathcal{SO}(3)$, there exists some deterministic $\mu \in \mathbb{R}^3$ such that

$$R^{\mathrm{ref}}\exp([\mu\times]) = R^{\mathrm{ref,post}} \tag{122}$$

Thus,

$$R^{\mathrm{ref}}\exp([\delta\times]) = R^{\mathrm{ref,post}}\exp\left(\left[\delta^{\mathrm{post}}\times\right]\right) \tag{123}$$

$$\Rightarrow \exp([\delta\times]) = \exp([\mu\times])\exp\left(\left[\delta^{\mathrm{post}}\times\right]\right) \tag{124}$$

Using (88),

$$[\delta\times] = [\mu\times] + \frac{-\mathrm{ad}_{[\mu\times]}}{\exp\left(-\mathrm{ad}_{[\mu\times]}\right)-1}\left(\left[\delta^{\mathrm{post}}\times\right]\right) + O\left(\left[\delta^{\mathrm{post}}\times\right]^2\right) \tag{125}$$

$$\Rightarrow \delta = \mu + \left[\frac{-\mathrm{ad}_{[\mu\times]}}{\exp\left(-\mathrm{ad}_{[\mu\times]}\right)-1}\left(\left[\delta^{\mathrm{post}}\times\right]\right) + O\left(\left[\delta^{\mathrm{post}}\times\right]^2\right)\right]^\vee \tag{126}$$

Throwing away higher order terms in δ^{post},

$$\delta \approx \mu + \frac{[\mu\times]}{I-\exp([-\mu\times])}\delta^{\mathrm{post}} \tag{127}$$

Thus,

$$\delta^{\text{post}} \approx \frac{I - \exp\left([-\mu\times]\right)}{[\mu\times]}(\delta - \mu) \tag{128}$$

$$= \Gamma(\mu)(\delta - \mu) \tag{129}$$

where (118) was used. The above is linear in the random variables δ and δ^{post}, therefore the constraint (29) implies that $\mu = \underset{(\delta)}{\mathrm{E}}[\delta]$.

D. Solution to Problem 1 when δ is two times the vector part of the unit-quaternion

Rearranging (124) and using quaternion algebra [82],

$$q(\delta^{\text{post}}) = q(-\mu) \odot q(\delta) \tag{130}$$

where

$$q : \left\{ x \in \mathbb{R}^3 \middle| \, \|x\| \le 2 \right\} \to \left\{ (a, v),\ a \in \mathbb{R}, v \in \mathbb{R}^3 \middle| \, |a|^2 + \|v\|^2 = 1 \right\} \tag{131}$$

$$x \mapsto \begin{bmatrix} a \\ v \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{4 - \|x\|^2} \\ x \end{bmatrix} \tag{132}$$

maps two times the vector part of the unit-quaternion to the unit-quaternion [23], and

$$q_1 \odot q_2 = (a_1, v_1) \odot (a_2, v_2) = \begin{bmatrix} a_1 a_2 - v_1^\top v_2 \\ a_1 v_2 + a_2 v_1 + v_1 \times v_2 \end{bmatrix} \tag{133}$$

Thus,

$$a_\delta^{\text{post}} = a_\mu a_\delta + \mu^\top \delta/4 \tag{134}$$

$$\delta^{\text{post}} = a_\mu \delta - a_\delta \mu + \delta \times \mu/2 \tag{135}$$

where $a_\mu = \sqrt{1 - \|\mu\|^2/4}$ and $a_\delta = \sqrt{1 - \|\delta\|^2/4}$. Continuing,

$$\delta^{\text{post}} = \frac{a_\mu^2 \delta - a_\mu a_\delta \mu}{a_\mu} - \mu \times \delta/2 \tag{136}$$

$$= \frac{a_\mu^2 \delta - (a_\delta^{\text{post}} - \mu^\top \delta/4)\mu}{a_\mu} - \mu \times \delta/2 \tag{137}$$

$$= \frac{a_\mu^2 \delta - \left(\sqrt{1 - \|\delta^{\text{post}}\|^2/4} - \mu^\top \delta/4\right)\mu}{a_\mu} - \mu \times \delta/2 \tag{138}$$

$$= \frac{a_\mu^2 \delta - \left(1 + O\left(\left\|\delta^{\text{post}}\right\|^2\right) - \mu^\top \delta/4\right)\mu}{a_\mu} - \mu \times \delta/2 \tag{139}$$

As in Appendix C, throwing away high-order terms in δ^{post},

$$\delta^{\text{post}} \approx \frac{a_\mu^2 \delta - \left(1 - \mu^\top \delta/4\right)\mu}{a_\mu} - \mu \times \delta/2 \tag{140}$$

$$= \frac{\left(1 - \|\mu\|^2/4\right)\delta - \left(1 - \mu^\top \delta/4\right)\mu}{\sqrt{1 - \|\mu\|^2/4}} - \mu \times \delta/2 \tag{141}$$

Using [74, Fact 3.5.25], $[\mu\times]^2 = \mu\mu^\top - \|\mu\|^2 I_{3\times3}$, and noting that $[\mu\times]\,\mu = 0$, the above becomes

$$\delta^{\text{post}} \approx \Gamma^{\text{qv}}(\mu)(\delta - \mu) \tag{142}$$

where $\Gamma^{\text{qv}}(\cdot)$ is given in (71). The above is linear in the random variables δ and δ^{post}, therefore the constraint (29) implies that $\mu = \underset{(\delta)}{\mathrm{E}}[\delta]$. The authors would like to acknowledge one of the reviewers for this result.

E. Computation of the Mean Attitude for the LG-UKF

Algorithm 2: Weighted intrinsic mean on $\mathcal{SO}(3)$.

input: Set of rotations $R^{(i)}$ with associated weights $w_m^{(i)}$, $i = 0, \ldots, n$, and an integer $N > 0$.
output: The weighted mean R^{ref}.

$R^{\text{ref}} \leftarrow R^{(0)}$;
for $j \leftarrow 0$ **to** N **do**
  $\Delta \leftarrow \sum_{i=0}^{n} w_m^{(i)} \log\left(R^{\text{ref}^{-1}} R^{(i)}\right)$;
  $R^{\text{ref}} \leftarrow R^{\text{ref}} \exp(\Delta)$;
end

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